II Semester M.Sc. Degree Examination, June/July 2014
(NS) (2006 Scheme)
MATHEMATICS
M-203 : Functional Analysis

Time : 3 Hours Max. Marks : 80

***Instructions** : 1) Answer **any five** questions. Choosing altleast **two** from **each** Part.*
*2) **All** questions carry **equal** marks.*

PART – A

1. a) Define :
   i) Normed linear space
   ii) Banach space.

   Give an example of a normed linear space which is not a Banach space. **4**

   b) Prove that a linear subspace of a Banach space is complete if and only if it is closed. **6**

   c) Define l_p space. Prove that l_p is a Banach space. **6**

2. a) If M is a closed linear subspace of a normed linear space X, then prove that the quotient space X/M is a normed linear space. Further show that X/M is a Banach space if X is a Banach space. **10**

   b) Show that any two norms on a finite dimensional normed linear space are equivalent. **6**

3. a) Show that a normed linear space is finite dimensional if and only if the closed unit ball is compact. **6**

   b) State and prove Hahn Banach theorem for the real care. **10**

4. a) State and prove open Mapping theorem. **9**

   b) Show that a normed linear space is separable if its dual space is separable. Is the converse true ? Justify. **7**

PART – B

5\. a) Define a Hilbert space. Prove that the inner product and norm in any Hilbert space H are continuous. **8**

b) In any Hilbert space H, prove that **8**

i) $|\langle x, y\rangle| \leq \|x\|\|y\|$,

ii) $\|x+y\| \leq \|x\|+\|y\|$ for all $x, y \in H$.

6\. a) If $f \in H^*$, show that there exists a Unique vector y in H such that $f(x) = \langle x, y \rangle$ for all x in H. **8**

b) Show that every non-zero Hilbert space contains a complete orthonormal set. **8**

7\. a) Let S, S_1, S_2 be non-empty subsets of a Hilbert space H, then prove the following

i) $\{0\}^{\perp} = H$ and $H^{\perp} = \{0\}$

ii) $S \cap S^{\perp} \leq \{0\}$

iii) $S_1 \subseteq S_2 \Rightarrow S_2^{\perp} \subseteq S_1^{\perp}$

iv) $S \subseteq S_1^{\perp\perp}$. **8**

b) Let H be a Hilbert space. Show $H = M \oplus M^{\perp}$. **8**

8\. a) Define self adjoint and normal operator on a Hilbert space H. Show that an operator T on H is

i) Self adjoint if and only if $\langle Tx, x \rangle$ is real for each $x \in H$,

ii) Normal if and only if $\|T^*x\| = \|Tx\|$ for each $x \in H$. **8**

b) If P is a projection on a Hilbert space H, with range M and null space N, then prove that $M \perp N$ if and only if P is self adjoint and in this care $N = M^{\perp}$. **4**

c) Prove that a closed linear subspace M of a Hilbert space H reduces an operator T if M is invariant under T and T^*. **4**